

An index ${}_2F_2$ hypergeometric transform

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Abstract

We construct a new one-parameter family of index hypergeometric transforms associated with the relativistic pseudoharmonic oscillator by using coherent states analysis.

1 Introduction

In [1], while introducing a class of coherent states attached to symmetric spaces of non-compact type, the authors have considered a family of weighted Bergman spaces labelled by a real parameter γ with $2\gamma = 1, 2, \dots$, as

$$\mathfrak{F}_\gamma(\mathbb{C}) = \left\{ \psi \text{ analytic on } \mathbb{C}, \int_{\mathbb{C}} |\psi(z)|^2 K_{\frac{1}{2}-\gamma}(2|z|) |z|^{2\gamma-1} d\mu(z) < \infty \right\} \quad (1.1)$$

where $K_\nu(\cdot)$ denotes the MacDonald function [2] and $d\mu(z)$ is the Lebesgue measure on \mathbb{C} .

In this paper, we construct a one-parameter family of index hypergeometric transforms mapping isometrically the space of complex-valued square integrable functions $\varphi(x)$ on positive real half-line onto spaces in (1.1) by

$$F_\gamma : L^2(\mathbb{R}_+, dx) \rightarrow \mathfrak{F}_\gamma(\mathbb{C})$$
$$F_\gamma[\varphi](z) = \frac{i^\gamma \exp(z)}{\Gamma(2\gamma)\Gamma(\gamma + \frac{1}{2})} \int_0^{+\infty} \frac{{}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; z\right)}{(2\gamma(\gamma - 1))^{ix} \Gamma(ix) \Gamma^{-2}(\gamma + ix)} \varphi(x) dx \quad (1.2)$$

with the Euler gamma function Γ and the ${}_2F_2$ hypergeometric series [3].

Our method is based on coherent state analysis. That is, we present a class of coherent states as superpositions of eigenstates of a relativistic model for the pseudoharmonic oscillator [4]. These eigenstates are expressed in terms of continuous dual Hahn polynomials [3]. In this superposition, the coefficients are

the basis elements of the weighted Bergman spaces in (1.1). The closed form of the constructed coherent states allows to define a coherent states transform which turns out to be the index hypergeometric transform in (1.2).

The paper is organized as follows. Section 2 deals with a brief formalism of coherent states and their corresponding coherent state transforms. In section 3, we summarize some needed tools on the relativistic pseudoharmonic oscillator. In section 4, we recall the definition of the weighted Bergman spaces we are dealing with. In section 5, we construct a class of coherent states from which we deduce a one-parameter family of index hypergeometric transforms.

2 Coherent states

Following [5], let (X, σ) be a measure space and let $\mathcal{A}_2(X) \subset L^2(X, \sigma)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=0}^\infty$ be an orthogonal basis of $\mathcal{A}_2(X)$ satisfying, for arbitrary $\xi \in X$,

$$\omega(\xi) := \sum_{n=0}^{\infty} \frac{|\Phi_n(\xi)|^2}{\rho_n} < +\infty, \quad (2.1)$$

where $\rho_n := \|\Phi_n\|_{L^2(X)}^2$. Define

$$\mathcal{K}(\xi, \zeta) := \sum_{n=0}^{\infty} \frac{\Phi_n(\xi) \overline{\Phi_n(\zeta)}}{\rho_n}, \quad \xi, \zeta \in X. \quad (2.2)$$

Then, $\mathcal{K}(\xi, \zeta)$ is a reproducing kernel, $\mathcal{A}_2(X)$ is the corresponding reproducing kernel Hilbert space and $\omega(\xi) = \mathcal{K}(\xi, \xi)$, $\xi \in X$.

Let \mathcal{H} be another Hilbert space with $\dim \mathcal{H} = \infty$ and $\{\phi_n\}_{n=0}^\infty$ be an orthonormal basis of \mathcal{H} . Therefore, define a coherent state as a ket vector $|\xi\rangle \in \mathcal{H}$ labelled by a point $\xi \in X$ as

$$|\xi\rangle := (\omega(\xi))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} |\phi_n\rangle. \quad (2.3)$$

We rewrite (2.3) using Dirac's bra-ket notation as

$$\langle x | \xi \rangle = (\omega(\xi))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \langle x | \phi_n \rangle. \quad (2.4)$$

By definition, it is straightforward to show that $\langle \xi | \xi \rangle = 1$ and the coherent state transform $W : \mathcal{H} \rightarrow \mathcal{A}_2(X) \subset L^2(X, \sigma)$ defined by

$$W[\phi](\xi) := (\omega(\xi))^{\frac{1}{2}} \langle \xi | \phi \rangle \quad (2.5)$$

is an isometry. Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$\langle \phi | \psi \rangle_{\mathcal{H}} = \langle W[\phi] | W[\psi] \rangle_{L^2(X)} = \int_X d\sigma(\xi) \omega(\xi) \langle \phi | \xi \rangle \langle \xi | \psi \rangle \quad (2.6)$$

and thereby we have a resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_X d\sigma(\xi) \omega(\xi) |\xi\rangle\langle\xi|, \quad (2.7)$$

where $\omega(\xi)$ appears as a weight function.

3 The weighted Bergman space $\mathfrak{F}_\gamma(\mathbb{C})$

As mentioned in Section.1, a countable set of Hilbert spaces whose elements are analytic functions on \mathbb{C} , have been introduced by the authors in ([1], p.51) as follows. For each fixed γ with $2\gamma = 1, 2, \dots$, the inner product is defined by

$$\langle \psi, \phi \rangle_\gamma := \int_{\mathbb{C}} \psi(z) \overline{\phi(z)} d\mu_\gamma(z), \quad (3.1)$$

where

$$d\mu_\gamma(z) = \frac{2}{\pi\Gamma(2\gamma)} \rho^{2\gamma-1} K_{\frac{1}{2}-\gamma}(2\rho) \rho d\theta d\rho, z = \rho e^{i\theta} \in \mathbb{C} \quad (3.2)$$

with the MacDonald function defined by ([5], p.183):

$$K_\nu(\rho) = \frac{1}{2} \left(\frac{\rho}{2}\right)^\nu \int_0^{+\infty} t^{-\nu-1} \exp\left(-t - \frac{\rho^2}{4t}\right) dt. \quad (3.3)$$

Using the notation in (3.1), the space in (1.1) also reads

$$\mathfrak{F}_\gamma(\mathbb{C}) = \left\{ \psi \text{ analytic on } \mathbb{C}, \langle \psi, \psi \rangle_\gamma < +\infty. \right\} \quad (3.4)$$

A well known orthonormal basis of the space in (3.4) is given by the functions

$$\psi_n^\gamma(z) := \frac{z^n}{\sqrt{n!(2\gamma)_n}}, \quad n = 0, 1, 2, \dots, z \in \mathbb{C}, \quad (3.5)$$

and its reproducing kernel is obtained as

$$\mathcal{K}_\gamma(z, w) := \sum_{n=0}^{+\infty} \frac{1}{(2\gamma)_n} \frac{(z\overline{w})^n}{n!} \quad (3.6)$$

in which $(a)_n$ is the Pochhammer symbol defined by $(a)_0 := 1$ and

$$(a)_n := a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (3.7)$$

One can use the formula for the modified Bessel function of the first kind ([5], p.77):

$$I_\nu(\zeta) = \sum_{n=0}^{+\infty} \frac{1}{n!\Gamma(\nu+n+1)} \left(\frac{\zeta}{2}\right)^{\nu+2n}, \quad (3.8)$$

for $\zeta = 2\sqrt{z\bar{w}}$ and $\nu = 2\gamma - 1$, to present (3.6) in a closed form as

$$\mathcal{K}_\gamma(z, w) = \Gamma(2\gamma) (z\bar{w})^{\frac{1}{2}-\gamma} I_{2\gamma-1}\left(2\sqrt{z\bar{w}}\right), \quad (3.9)$$

So that the diagonal function of (3.9) reads

$$\mathcal{K}_\gamma(z, z) = \Gamma(2\gamma) |z|^{1-2\gamma} I_{2\gamma-1}(2|z|), z \in \mathbb{C}. \quad (3.10)$$

The latter will be used in the sequel.

4 A relativistic model for the pseudoharmonic oscillator

In this section, we recall some needed results which have been developed in [4] where the authors considered a model for the relativistic pseudoharmonic oscillator with the following interaction potential

$$V(x) := \left(\frac{1}{2} m \omega^2 x(x + i\lambda) + \frac{g}{x(x + i\lambda)} \right) e^{i\lambda \partial_x} \quad (4.1)$$

where ω is a frequency, $g \geq 0$ is a real quantity and $\lambda = \hbar/mc$ denotes the Compton wavelength defined by the ratio of Planck's constant \hbar by the mass m times the speed of light c .

The corresponding stationary Schrödinger equation is described by the finite-difference equation

$$\left(mc^2 \cosh i\lambda \partial_x + \frac{1}{2} m \omega^2 x(x + i\lambda) e^{i\lambda \partial_x} + \frac{g}{x(x + i\lambda)} e^{i\lambda \partial_x} \right) \varphi(x) = \epsilon \varphi(x) \quad (4.2)$$

with the boundary conditions for the wave function $\varphi(0) = 0$ and $\varphi(\infty) = 0$. The energy spectrum of the Schrödinger operator in (4.2) is

$$\epsilon_n := \hbar \omega (2n + \alpha_+ + \alpha_-), n = 0, 1, 2, \dots \quad (4.3)$$

where

$$\alpha_\pm = \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2}{\omega_0} \left(1 \pm \sqrt{1 - 8g_0 \omega_0^2} \right)} \quad (4.4)$$

and

$$\omega_0 = \frac{\hbar \omega}{mc^2}, \quad g_0 = \frac{mg}{\hbar^2} \quad (4.5)$$

The corresponding orthonormalized eigenstates of the form

$$\langle x | \varphi_n^{\alpha_\pm} \rangle = c_n (-x)^{(\alpha_+)} \omega_0^{ix} \Gamma(\alpha_- + ix) S_n \left(x^2; \alpha_+, \alpha_-, \frac{1}{2} \right) \quad (4.6)$$

where

$$c_n = \left(2^{-1} n! \Gamma(n + \alpha_+ + \alpha_-) \Gamma\left(n + \alpha_+ + \frac{1}{2}\right) \Gamma\left(n + \alpha_- + \frac{1}{2}\right) \right)^{-\frac{1}{2}} \quad (4.7)$$

and

$$S_n(x^2; a, b, c) := (a+b)_n (a+c)_n {}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1\right) \quad (4.8)$$

is the continuous dual Hahn polynomial ([3], p.331) given in terms of the ${}_3F_2$ -sum. The notation

$$\zeta^{(\nu)} := i^\nu \frac{\Gamma(\nu - i\zeta)}{\Gamma(-i\zeta)} \quad (4.9)$$

means the generalized degree ([4],[6]). The wavefunctions in (4.6) satisfy the relations :

$$\int_0^{+\infty} \langle x | \varphi_m^{\alpha_\pm} \rangle \overline{\langle x | \varphi_n^{\alpha_\pm} \rangle} dx = \delta_{n,m} \quad (4.10)$$

which means that they constitute an orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+, dx)$.

5 An index hypergeometric transform

We now combine the two basis $(\psi_n^\gamma)_n$ in (3.5) and $(|\varphi_n^{\alpha_\pm}\rangle)_n$ in (4.6) together with the function $\mathcal{K}_\gamma(.,.)$ in (3.10) to construct for every fixed $\gamma \in \mathbb{R}$ with $2\gamma = 1, 2, \dots$, a set of coherent states $(|z; \gamma\rangle)_{z \in \mathbb{C}}$ labelled by points $z \in \mathbb{C}$ according to definition (2.3) by setting

$$|z; \gamma\rangle := (\mathcal{K}_\gamma(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n n!}} |\varphi_n^{\alpha_\pm}\rangle \quad (5.1)$$

For our purpose, we choose the parameters g and ω in (4.1) are such that $8g\omega^2 = mc^4$. In such case and in view of (4.5), we get that

$$1 - 8g_0\omega_0^2 = 0 \quad (5.2)$$

and Eq.(4.4) reduces to

$$\alpha_+ = \alpha_- = \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2}{\omega_0}} \quad (5.3)$$

Now, if we set

$$\gamma := \alpha_+ = \alpha_- \quad (5.4)$$

which means that

$$2\gamma - 1 = \sqrt{1 + \frac{16g\omega}{\hbar c^2}} = \sqrt{1 + 2\frac{mc^2}{\hbar\omega}}. \quad (5.5)$$

Then, we can establish a closed form for wavefunctions of the coherent states defined in (5.1) as follows. Let $x \in \mathbb{R}_+$. Then, starting from (5.1) with the condition (5.4), we can write successively

$$\langle x | z; \gamma \rangle = (\mathcal{K}_\gamma(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n} n!} \langle x | n; \gamma \rangle, \quad (5.6)$$

$$= (\mathcal{K}_\gamma(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{c_n z^n}{\sqrt{(2\gamma)_n} n!} (-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix) S_n\left(x^2; \gamma, \gamma, \frac{1}{2}\right) \quad (5.7)$$

$$= \frac{(-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\mathcal{K}_\gamma(z, z)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n} n!} c_n S_n\left(x^2; \gamma, \gamma, \frac{1}{2}\right) \quad (5.8)$$

$$= \frac{(-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\mathcal{K}_\gamma(z, z)}} \sum_{n=0}^{\infty} \frac{S_n\left(x^2; \gamma, \gamma, \frac{1}{2}\right)}{\sqrt{(2\gamma)_n} \sqrt{\Gamma(2\gamma + n)} \Gamma(n + \gamma + \frac{1}{2})} \frac{z^n}{n!} \quad (5.9)$$

Let us set

$$\langle x | z; \gamma \rangle = \frac{(-x)^{(\alpha)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\omega_\gamma(z)}} \mathfrak{G}_\gamma(x, z) \quad (5.10)$$

where

$$\mathfrak{G}_\gamma(x, z) := \sum_{n=0}^{\infty} \frac{S_n\left(x^2; \gamma, \gamma, \frac{1}{2}\right)}{\sqrt{(2\gamma)_n} \sqrt{\Gamma(2\gamma + n)} \Gamma(n + \gamma + \frac{1}{2})} \frac{z^n}{n!} \quad (5.11)$$

$$= \frac{1}{\sqrt{\Gamma(2\gamma)} \Gamma(\gamma + \frac{1}{2})} \sum_{n=0}^{\infty} \frac{S_n\left(x^2; \gamma, \gamma, \frac{1}{2}\right)}{(2\gamma)_n (\gamma + \frac{1}{2})_n} \frac{z^n}{n!} \quad (5.12)$$

Making use of the following generating formula for the continuous dual Hahn ([3], p.349):

$$e^\xi {}_2F_2\left(\begin{matrix} a + ix, a - ix \\ a + b, a + c \end{matrix}; -\xi\right) = \sum_{n=0}^{+\infty} \frac{S_n(x^2; a, b, c)}{(a + b)_n (a + c)_n} \frac{\xi^n}{n!} \quad (5.13)$$

for $a = b = \gamma$, $c = \frac{1}{2}$ and $\xi = z$, we obtain that

$$\mathfrak{G}_\gamma(x, z) = \frac{e^z}{\sqrt{\Gamma(2\gamma)} \Gamma(\gamma + \frac{1}{2})} {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right). \quad (5.14)$$

Returning back to (5.10) and inserting (5.14), we get that

$$\langle x | z; \gamma \rangle = \frac{(-x)^{(\alpha)} \omega_0^{ix} \Gamma(\gamma + ix) e^z}{\sqrt{\mathcal{K}_\gamma(z, z)}} \frac{{}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right)}{\sqrt{\Gamma(2\gamma)} \Gamma(\gamma + \frac{1}{2})} \quad (5.15)$$

Next, making use of (4.9), we write $(-x)^{(\gamma)}$ in (5.15) as

$$(-x)^{(\gamma)} = i^\gamma \frac{\Gamma(\gamma + ix)}{\Gamma(ix)} \quad (5.16)$$

On the other hand, by (5.3) and (5.4), we can write ω_0 in terms of γ as

$$\omega_0 = \frac{1}{\gamma(2\gamma-1)} \quad (5.17)$$

Now, summarizing up the above calculation, Eq.(5.15) takes the form

$$\langle x | z; \gamma \rangle = \frac{i^\gamma \frac{\Gamma(\gamma+ix)}{\Gamma(ix)} \left(\frac{1}{2\gamma(\gamma-1)} \right)^{ix} \Gamma(\gamma+ix) e^z {}_2F_2 \left(\begin{matrix} \gamma+ix, \gamma-ix \\ 2\gamma, \gamma+\frac{1}{2} \end{matrix}; -z \right)}{\sqrt{\mathcal{K}_\gamma(z, z)} \sqrt{\Gamma(2\gamma)\Gamma(\gamma+\frac{1}{2})}} \quad (5.18)$$

Finally, making use of Eq. (3.10), arrive at the following closed form for

$$\langle x | z; \gamma \rangle = \frac{i^\gamma (2\gamma(\gamma-1))^{-ix} \Gamma^2(\gamma+ix) e^z {}_2F_2 \left(\begin{matrix} \gamma+ix, \gamma-ix \\ 2\gamma, \gamma+\frac{1}{2} \end{matrix}; -z \right)}{\sqrt{|z|^{1-2\gamma} I_{2\gamma-1}(2|z|) \Gamma(2\gamma) \Gamma(\gamma+\frac{1}{2}) \Gamma(ix)}} \quad (5.19)$$

Finally, by (2.5), the coherent state transform corresponding the coherent states in (5.6) with the closed form (5.19) of their wavefunctions is the isometry mapping

$$F_\gamma : L^2(\mathbb{R}^+, dx) \rightarrow \mathfrak{F}_\gamma(\mathbb{C}) \quad (5.20)$$

defined by

$$F_\gamma[\varphi](z) := (\mathcal{K}_\gamma(z, z))^{\frac{1}{2}} \langle z; \gamma | \varphi \rangle_{L^2(\mathbb{R}^+, dx)} \quad (5.21)$$

Explicitly,

$$F_\gamma[\varphi](z) = \frac{i^\gamma \exp(z)}{\Gamma(2\gamma)\Gamma(\gamma+\frac{1}{2})} \int_0^{+\infty} \frac{\Gamma^2(\gamma+ix) {}_2F_2 \left(\begin{matrix} \gamma+ix, \gamma-ix \\ 2\gamma, \gamma+\frac{1}{2} \end{matrix}; -z \right)}{(2\gamma(\gamma-1))^{ix} \Gamma(ix)} \overline{\varphi(x)} dx \quad (5.22)$$

which is the announced transform in (1.2).

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AN INDEX ${}_2F_2$ HYPERGEOMETRIC TRANSFORM

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1 INTRODUCTION

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where $K_\nu(\cdot)$ denotes the MacDonald function [2] and $d\mu(z)$ is the Lebesgue measure on \mathbb{C} .

In this paper, we construct a one-parameter family of index hypergeometric transforms mapping isometrically the space of complex-valued square integrable functions $\varphi(x)$ on positive real half-line onto spaces in (1.1) by

$$F_\gamma : L^2(\mathbb{R}_+, dx) \rightarrow \mathfrak{F}_\gamma(\mathbb{C})$$

$$F_\gamma[\varphi](z) = \frac{i^\gamma \exp(z)}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} \int_0^{+\infty} \frac{{}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; z\right)}{(2\gamma(\gamma-1))^{ix} \Gamma(ix) \Gamma^{-2}(\gamma + ix)} \varphi(x) dx \quad (1.2)$$

with the Euler gamma function Γ and the ${}_2F_2$ hypergeometric series [3].

Our method is based on coherent state analysis. That is, we present a class of coherent states as superpositions of eigenstates of a relativistic model for the pseudoharmonic oscillator [4]. These eigenstates are expressed in terms of continuous dual Hahn polynomials [3]. In this superposition, the coefficients are the basis elements of the weighted Bergman spaces in (1.1). The closed form of the constructed coherent states allows to define a coherent states transform which turns out to be the index hypergeometric transform in (1.2).

The paper is organized as follows. Section 2 deals with a brief formalism of coherent states and their corresponding coherent state transforms. In section 3, we summarize some needed tools on the relativistic pseudoharmonic oscillator. In section 4, we recall the definition of the weighted Bergman spaces we are dealing with. In section 5, we construct a class of coherent states from which we deduce a one-parameter family of index hypergeometric transforms.

2 COHERENT STATES

Following [5, pp.72-76], let (X, σ) be a measure space and let $\mathcal{A}_2(x) \subset L^2(X, \sigma)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=0}^\infty$ be an orthogonal basis of $\mathcal{A}_2(x)$ satisfying, for arbitrary $\xi \in X$,

$$\omega(\xi) := \sum_{n=0}^{\infty} \frac{|\Phi_n(\xi)|^2}{\rho_n} < +\infty, \quad (2.1)$$

where $\rho_n := \|\Phi_n\|_{L^2(X)}^2$. Define

$$\mathcal{K}(\xi, \zeta) := \sum_{n=0}^{\infty} \frac{\Phi_n(\xi) \overline{\Phi_n(\zeta)}}{\rho_n}, \quad \xi, \zeta \in X. \quad (2.2)$$

Then, $\mathcal{K}(\xi, \zeta)$ is a reproducing kernel, $\mathcal{A}_2(x)$ is the corresponding reproducing kernel Hilbert space and $\omega(\xi) = \mathcal{K}(\xi, \xi)$, $\xi \in X$.

Let \mathcal{H} be another Hilbert space with $\dim \mathcal{H} = \infty$ and $\{\phi_n\}_{n=0}^\infty$ be an orthonormal basis of \mathcal{H} . Therefore, define a coherent state as a ket vector $|\xi\rangle \in \mathcal{H}$ labelled by a point $\xi \in X$ as

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We rewrite (2.3) using Dirac's bra-ket notation as

$$\langle x | \xi \rangle = (\omega(\xi))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \langle x | \phi_n \rangle. \quad (2.4)$$

By definition, it is straightforward to show that $\langle \xi | \xi \rangle = 1$ and the coherent state transform $W : \mathcal{H} \rightarrow \mathcal{A}_2(x) \subset L^2(X, \sigma)$ defined by

$$W[\phi](\xi) := (\omega(\xi))^{\frac{1}{2}} \langle \xi | \phi \rangle \quad (2.5)$$

is an isometry. Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$\langle \phi | \psi \rangle_{\mathcal{H}} = \langle W[\phi] | W[\psi] \rangle_{L^2(X)} = \int_X d\sigma(\xi) \omega(\xi) \langle \phi | \xi \rangle \langle \xi | \psi \rangle \quad (2.6)$$

and thereby we have a resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_X d\sigma(\xi) \omega(\xi) |\xi\rangle \langle \xi|, \quad (2.7)$$

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As mentioned in Section.1, a countable set of Hilbert spaces whose elements are analytic functions on \mathbb{C} , have been introduced by the authors in ([1, p.51]) as follows. For each fixed γ with $2\gamma = 1, 2, \dots$, the inner product is defined by

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where

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with the MacDonald function defined by ([5, p.183]):

$$K_\nu(\rho) = \frac{1}{2} \left(\frac{\rho}{2}\right)^\nu \int_0^{+\infty} t^{-\nu-1} \exp\left(-t - \frac{\rho^2}{4t}\right) dt. \quad (3.3)$$

Using the notation in (3.1), the space in (1.1) also reads

$$\mathfrak{F}_\gamma(\mathbb{C}) = \left\{ \psi \text{ analytic on } \mathbb{C}, \langle \psi, \psi \rangle_\gamma < +\infty. \right\} \quad (3.4)$$

A well known orthonormal basis of the space in (3.4) is given by the functions

$$\psi_n^\gamma(z) := \frac{z^n}{\sqrt{n! (2\gamma)_n}}, \quad n = 0, 1, 2, \dots, z \in \mathbb{C}, \quad (3.5)$$

and its reproducing kernel is obtained as

$$\mathcal{K}_\gamma(z, w) := \sum_{n=0}^{+\infty} \frac{1}{(2\gamma)_n} \frac{(z\bar{w})^n}{n!} \quad (3.6)$$

in which $(a)_n$ is the Pochhammer symbol defined by $(a)_0 := 1$ and

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One can use the formula for the modified Bessel function of the first kind ([5, p.77]):

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for $\zeta = 2\sqrt{z\bar{w}}$ and $\nu = 2\gamma - 1$, to present (3.6) in a closed form as

$$\mathcal{K}_\gamma(z, w) = \Gamma(2\gamma) (z\bar{w})^{\frac{1}{2}-\gamma} I_{2\gamma-1}\left(2\sqrt{z\bar{w}}\right), \quad (3.9)$$

So that the diagonal function of (3.9) reads

$$\mathcal{K}_\gamma(z, z) = \Gamma(2\gamma) |z|^{1-2\gamma} I_{2\gamma-1}(2|z|), \quad z \in \mathbb{C}. \quad (3.10)$$

The latter will be used in the sequel.

4 A RELATIVISTIC MODEL FOR THE PSEUDOHARMONIC OSCILLATOR

In this section, we recall some needed results which have been developed in [4] where the authors considered a model for the relativistic pseudoharmonic oscillator with the following interaction potential

$$V(x) := \left(\frac{1}{2} m \omega^2 x (x + i\lambda) + \frac{g}{x(x + i\lambda)} \right) e^{i\lambda \partial_x} \quad (4.1)$$

where ω is a frequency, $g \geq 0$ is a real quantity and $\lambda = \hbar/mc$ denotes the Compton wavelength defined by the ratio of Planck's constant \hbar by the mass m times the speed of light c .

The corresponding stationary Schrödinger equation is described by the finite-difference equation

$$\left(mc^2 \cosh i\lambda \partial_x + \frac{1}{2} m\omega^2 x (x + i\lambda) e^{i\lambda \partial_x} + \frac{g}{x(x + i\lambda)} e^{i\lambda \partial_x} \right) \varphi(x) = \epsilon \varphi(x) \quad (4.2)$$

with the boundary conditions for the wave function $\varphi(0) = 0$ and $\varphi(\infty) = 0$. The energy spectrum of the Schrödinger operator in (4.2) is

$$\epsilon_n := \hbar\omega (2n + \alpha_+ + \alpha_-), n = 0, 1, 2, \dots \quad (4.3)$$

where

$$\alpha_{\pm} = \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2}{\omega_0} \left(1 \pm \sqrt{1 - 8g_0\omega_0^2} \right)} \quad (4.4)$$

and

$$\omega_0 = \frac{\hbar\omega}{mc^2}, \quad g_0 = \frac{mg}{\hbar^2} \quad (4.5)$$

The corresponding orthonormalized eigenstates of the form

$$\langle x | \varphi_n^{\alpha_{\pm}} \rangle = c_n (-x)^{(\alpha_+)} \omega_0^{ix} \Gamma(\alpha_- + ix) S_n \left(x^2; \alpha_+, \alpha_-, \frac{1}{2} \right) \quad (4.6)$$

where

$$c_n = \left(2^{-1} n! \Gamma(n + \alpha_+ + \alpha_-) \Gamma \left(n + \alpha_+ + \frac{1}{2} \right) \Gamma \left(n + \alpha_- + \frac{1}{2} \right) \right)^{-\frac{1}{2}} \quad (4.7)$$

and

$$S_n \left(x^2; a, b, c \right) := (a + b)_n (a + c)_n {}_3F_2 \left(\begin{matrix} -n, a + ix, a - ix \\ a + b, a + c \end{matrix}; 1 \right) \quad (4.8)$$

is the continuous dual Hahn polynomial ([3, p.331]) given in terms of the ${}_3F_2$ -sum. The notation

$$\zeta^{(\nu)} := i^{\nu} \frac{\Gamma(\nu - i\zeta)}{\Gamma(-i\zeta)} \quad (4.9)$$

means the generalized degree ([4, 6]). The wavefunctions in (4.6) satisfy the relations:

$$\int_0^{+\infty} \langle x | \varphi_m^{\alpha_{\pm}} \rangle \overline{\langle x | \varphi_n^{\alpha_{\pm}} \rangle} dx = \delta_{n,m} \quad (4.10)$$

which means that they constitute an orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+, dx)$.

5 AN INDEX HYPERGEOMETRIC TRANSFORM

We now combine the two basis $(\psi_n^{\gamma})_n$ in (3.5) and $(|\varphi_n^{\alpha_{\pm}}\rangle)_n$ in (4.6) together with the function $\mathcal{K}_{\gamma}(\cdot, \cdot)$ in (3.10) to construct for every fixed $\gamma \in \mathbb{R}$ with $2\gamma = 1, 2, \dots$, a set of coherent states $(|z; \gamma\rangle)_{z \in \mathbb{C}}$ labelled by points $z \in \mathbb{C}$ according to definition (2.3) by setting

$$|z; \gamma\rangle := (\mathcal{K}_{\gamma}(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n n!}} |\varphi_n^{\alpha_{\pm}}\rangle. \quad (5.1)$$

For our purpose, we choose the parameters g and ω in (4.1) are such that $8g\omega^2 = mc^4$. In such case and in view of (4.5), we get that

$$1 - 8g_0\omega_0^2 = 0 \quad (5.2)$$

and Eq.(4.4) reduces to

$$\alpha_+ = \alpha_- = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{\omega_0}} \quad (5.3)$$

Now, if we set

$$\gamma := \alpha_+ = \alpha_- \quad (5.4)$$

which means that

$$2\gamma - 1 = \sqrt{1 + \frac{16g\omega}{\hbar c^2}} = \sqrt{1 + 2\frac{mc^2}{\hbar\omega}}. \quad (5.5)$$

Then, we can establish a closed form for wavefunctions of the coherent states defined in (5.1) as follows. Let $x \in \mathbb{R}_+$. Then, starting from (5.1) with the condition (5.4), we can write successively

$$\langle x | z; \gamma \rangle = (\mathcal{K}_\gamma(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n} n!} \langle x | n; \gamma \rangle, \quad (5.6)$$

$$= (\mathcal{K}_\gamma(z, z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{c_n z^n}{\sqrt{(2\gamma)_n} n!} (-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix) S_n \left(x^2; \gamma, \gamma; \frac{1}{2} \right) \quad (5.7)$$

$$= \frac{(-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\mathcal{K}_\gamma(z, z)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\gamma)_n} n!} c_n S_n \left(x^2; \gamma, \gamma; \frac{1}{2} \right) \quad (5.8)$$

$$= \frac{(-x)^{(\gamma)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\mathcal{K}_\gamma(z, z)}} \sum_{n=0}^{\infty} \frac{S_n \left(x^2; \gamma, \gamma; \frac{1}{2} \right)}{\sqrt{(2\gamma)_n} \sqrt{\Gamma(2\gamma + n)} \Gamma \left(n + \gamma + \frac{1}{2} \right)} \frac{z^n}{n!} \quad (5.9)$$

Let us set

$$\langle x | z; \gamma \rangle = \frac{(-x)^{(\alpha)} \omega_0^{ix} \Gamma(\gamma + ix)}{\sqrt{\omega_\gamma(z)}} \mathfrak{G}_\gamma(x, z) \quad (5.10)$$

where

$$\mathfrak{G}_\gamma(x, z) := \sum_{n=0}^{\infty} \frac{S_n \left(x^2; \gamma, \gamma; \frac{1}{2} \right)}{\sqrt{(2\gamma)_n} \sqrt{\Gamma(2\gamma + n)} \Gamma \left(n + \gamma + \frac{1}{2} \right)} \frac{z^n}{n!} \quad (5.11)$$

$$= \frac{1}{\sqrt{\Gamma(2\gamma)} \Gamma \left(\gamma + \frac{1}{2} \right)} \sum_{n=0}^{\infty} \frac{S_n \left(x^2; \gamma, \gamma; \frac{1}{2} \right)}{(2\gamma)_n \left(\gamma + \frac{1}{2} \right)_n} \frac{z^n}{n!} \quad (5.12)$$

Making use of the following generating formula for the continuous dual Hahn ([3, p.349]):

$$e^{\xi} {}_2F_2 \left(\begin{matrix} a + ix, a - ix \\ a + b, a + c \end{matrix} ; -\xi \right) = \sum_{n=0}^{+\infty} \frac{S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n} \frac{\xi^n}{n!} \quad (5.13)$$

for $a = b = \gamma, c = \frac{1}{2}$ and $\xi = z$, we obtain that

$$\mathfrak{G}_\gamma(x, z) = \frac{e^z}{\sqrt{\Gamma(2\gamma)}\Gamma\left(\gamma + \frac{1}{2}\right)} {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right). \quad (5.14)$$

Returning back to (5.10) and inserting (5.14), we get that

$$\langle x | z; \gamma \rangle = \frac{(-x)^{(\alpha)} \omega_0^{ix} \Gamma(\gamma + ix) e^z {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right)}{\sqrt{\mathcal{K}_\gamma(z, z)} \sqrt{\Gamma(2\gamma)}\Gamma\left(\gamma + \frac{1}{2}\right)}. \quad (5.15)$$

Next, making use of (4.9), we write $(-x)^{(\gamma)}$ in (5.15) as

$$(-x)^{(\gamma)} = i^\gamma \frac{\Gamma(\gamma + ix)}{\Gamma(ix)} \quad (5.16)$$

On the other hand, by (5.3) and (5.4), we can write ω_0 in terms of γ as

$$\omega_0 = \frac{1}{2\gamma(\gamma - 1)} \quad (5.17)$$

Now, summarizing up the above calculation, Eq.(5.15) takes the form

$$\langle x | z; \gamma \rangle = \frac{i^\gamma \frac{\Gamma(\gamma + ix)}{\Gamma(ix)} \left(\frac{1}{2\gamma(\gamma - 1)}\right)^{ix} \Gamma(\gamma + ix) e^z {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right)}{\sqrt{\mathcal{K}_\gamma(z, z)} \sqrt{\Gamma(2\gamma)}\Gamma\left(\gamma + \frac{1}{2}\right)} \quad (5.18)$$

Finally, making use of Eq. (3.10), arrive at the following closed form for

$$\langle x | z; \gamma \rangle = \frac{i^\gamma (2\gamma(\gamma - 1))^{-ix} \Gamma^2(\gamma + ix) e^z {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right)}{\sqrt{|z|^{1-2\gamma} I_{2\gamma-1}(2|z|)} \Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma(ix)} \quad (5.19)$$

Finally, by (2.5), the coherent state transform corresponding the coherent states in (5.6) with the closed form (5.19) of their wavefunctions is the isometry mapping

$$F_\gamma : L^2(\mathbb{R}^+, dx) \rightarrow \mathfrak{F}_\gamma(\mathbb{C}) \quad (5.20)$$

defined by

$$F_\gamma[\varphi](z) := (\mathcal{K}_\gamma(z, z))^{\frac{1}{2}} \langle z; \gamma | \varphi \rangle_{L^2(\mathbb{R}^+, dx)} \quad (5.21)$$

Explicitly,

$$F_\gamma[\varphi](z) = \frac{i^\gamma \exp(z)}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} \int_0^{+\infty} \frac{\Gamma^2(\gamma + ix) {}_2F_2\left(\begin{matrix} \gamma + ix, \gamma - ix \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix}; -z\right)}{(2\gamma(\gamma - 1))^{ix} \Gamma(ix)} \overline{\varphi(x)} dx \quad (5.22)$$

which is the announced transform in (1.2).

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